

# Non-Fermi liquids from holography

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based on:

Hong Liu, JM, David Vegh, 0903.2477

Tom Faulkner, HL, JM, DV, 0907.2694

TF, Nabil Iqbal, HL, JM, DV, in progress

see also: Sung-Sik Lee, 0809.3402

Cubrovic, Zaanen, Schalm, 0904.1933

# Outline

## Introduction: Fermions at finite density

- Theoretical ubiquity of Landau fermi liquid

- Existence of non-Fermi liquids

## Holographic duality

- Lightning review

- Sample application

## Charged AdS black holes and frustration

## Fermion green functions, numerically

## Analytic understanding of Fermi surface behavior

## Concluding remarks

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## Fermions at finite density

The **metallic** states of a finite density of fermions that we understand are described by Landau Fermi liquid theory: a free RG fixed point, stable [e.g. Benfatto-Gallivotti, Polchinski, Shankar, 1992] (modulo BCS instability)

Landau quasiparticles  $\rightarrow$  poles in single-fermion green function

$$G_R(t, \vec{x}) = i\theta(t) \langle \{ \psi^\dagger(t, \vec{x}), \psi(0, \vec{0}) \} \rangle_\mu$$

$$G_R(\omega, k) = \frac{Z}{\omega - v_F k_\perp + i\Gamma} + \dots, \quad k_\perp \equiv |\vec{k}| - k_F$$

Landau quasiparticles are long-lived: width is  $\Gamma \sim \omega_\star^2$ .

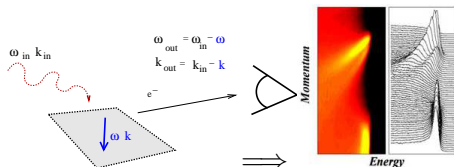
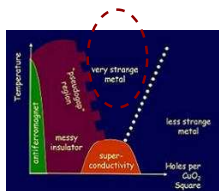
they only interact by irrelevant operators  
residue  $Z$  is finite on Fermi surface.

$$\text{spectral density : } A(\omega, k) \equiv \frac{1}{\pi} \text{Im } G_R(\omega, k) \xrightarrow{k_\perp \rightarrow 0} Z \delta(\omega - v_F k_\perp)$$

Reliable calculation of thermodynamics and transport relies on this.

# Non-Fermi liquids exist but are mysterious

For example: normal phase of optimally-doped cuprates:



[ZX Shen]

among other anomalies: ARPES shows gapless modes at finite  $k$  with width  $\Gamma(\omega_*) \sim \omega_*$ , vanishing residue  $Z^{k_{\perp} \rightarrow 0} \rightarrow 0$ .

Working definition:

Still a sharp Fermi surface      nonanalyticity of  $A(\omega \sim 0, k \sim k_F)$   
but no long-lived quasiparticles.

Note: we're not going to worry about angle-dependence!

# My understanding of the theoretical status of NFL

- Luttinger liquid (1+1-d) ✓
- numerics on Hubbard model
- high-temperature classical regime
- loophole in RG argument:

couple a Landau FL **perturbatively** to a bosonic mode  
(magnetic photon, slave-boson gauge field, ferromagnetism, SDW, Pomeranchuk order parameter...)

[Holstein et al, Baym et al, Halperin-Lee-Read, Polchinski, Altshuler-Ioffe-Millis, Nayak-Wilczek, Schafer-Schwenzer, Chubukov et al, Metzner et al, S-S Lee...]

→ nonanalytic behavior in  $G^R(\omega) \equiv \frac{1}{v_F k_{\perp} + \Sigma(\omega, k)}$  at FS:

$$\Sigma(\omega) \sim \begin{cases} \omega^{2/3} & d = 2 + 1 \\ \omega \log \omega & d = 3 + 1 \end{cases} \implies Z \xrightarrow{k_{\perp} \rightarrow 0} 0, \quad \frac{\Gamma(k_{\perp})}{\omega_{*}(k_{\perp})} \xrightarrow{k_{\perp} \rightarrow 0} \text{const}$$

**Comment:** non-analytic terms  $\propto$  control parameter  $\implies$   
perturbative answer is parametrically reliable  $\leftrightarrow$

effect is visible only at parametrically low temperatures.

It would be valuable to have a non-perturbative description of such a phase in more than one dimension.

## **Gravity dual?**

We're not going to look for a gravity dual of the whole material.  
Rather: lessons for principles of NFL.

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# holographic duality (AdS/CFT)

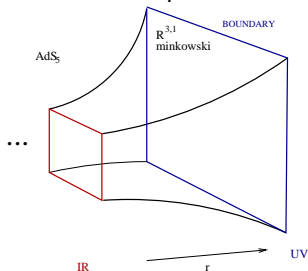
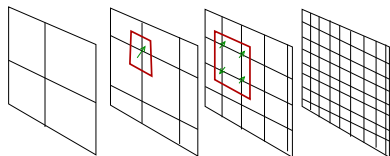
[Maldacena; Witten; Gubser-Klebanov-Polyakov]

gravity in  $AdS_{d+1} = d$ -dimensional Conformal Field Theory

(many generalizations, CFT is best-understood.)

isometries of  $AdS_{d+1} \Leftrightarrow$  conformal symmetry

$$AdS : ds^2 = \frac{r^2}{R^2} (-dt^2 + d\vec{x}^2) + R^2 \frac{dr^2}{r^2}$$



The extra ('radial') dimension is the resolution scale.

the bulk picture is a hologram:

excitations with different wavelengths get put in different places

## when is it useful?

$$\begin{aligned} Z_{QFT}[\text{sources}] &= Z_{\text{quantum gravity}}[\text{boundary conditions at } r \rightarrow \infty] \\ &\approx e^{-N^2 S_{\text{bulk}}[\text{boundary conditions at } r \rightarrow \infty]} \Big|_{\text{extremum of } S_{\text{bulk}}} \end{aligned}$$

classical gravity (sharp saddle)  $\Leftrightarrow$  many degrees of freedom per point,  $N^2 \gg 1$

fields in  $AdS_{d+1}$   $\Leftrightarrow$  operators in CFT  
mass  $\Leftrightarrow$  scaling dimension

boundary conditions on bulk fields  $\Leftrightarrow$  couplings in field theory

e.g.: boundary value of bulk metric  $\lim_{r \rightarrow \infty} g_{\mu\nu}$   
= source for stress-energy tensor  $T^{\mu\nu}$

different couplings in bulk action  $\Leftrightarrow$  different field theories

large  $AdS$  radius  $R$   $\Leftrightarrow$  strong coupling of QFT

# confidence-building measures

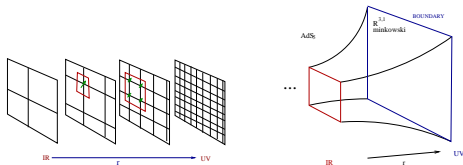
- ▶ 1. **Many** detailed checks in special examples  
examples: relativistic gauge theories (fields are  $N \times N$  matrices), with extra symmetries (conformal invariance, supersymmetry)  
checks: 'BPS quantities,' integrable techniques, some numerics
- ▶ 2. sensible answers for physics questions  
rediscoveries of known physical phenomena: e.g. color confinement, chiral symmetry breaking, thermo, hydro, thermal screening, entanglement entropy, chiral anomalies, superconductivity, ...  
Gravity limit, when valid, says who are the correct variables.  
Answers questions about thermodynamics, transport, RG flow, ... in terms of geometric objects.
- ▶ 3. applications to quark-gluon plasma (QGP)  
benchmark for viscosity, hard probes of medium, approach to equilibrium

# simple pictures for hard problems, an example

Bulk geometry is a spectrograph separating the theory by energy scales.

$$ds^2 = w(r)^2 (-dt^2 + d\vec{x}^2) + R^2 \frac{dr^2}{r^2}$$

CFT: bulk geometry goes on forever, warp factor  $w(r) = \frac{r^2}{R^2} \rightarrow 0$ :

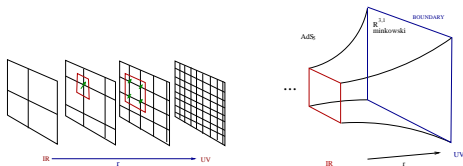


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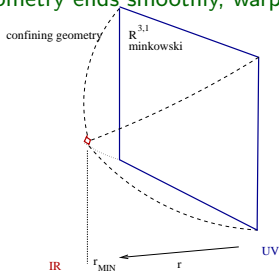
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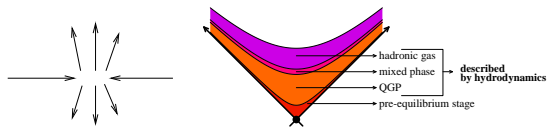
Model with a gap: geometry ends smoothly, warp factor  $w(r)$  has a min:



## Sample application: approach to equilibrium

Important question for interpreting RHIC data: how long does it take before hydro sets in?

initially in gold-gold collision: anisotropic momentum-space distribution



[Heller-Janik-Peschanski]

after time  $\tau_{th}$ : locally thermal distribution and hydrodynamics.

At RHIC:  $\tau_{th}$  much smaller than perturbation theory answer.

( $\tau_{th}$  affects measurement of viscosity:

good elliptic flow requires both low  $\eta$  and early applicability of hydro)

Thermal equilibrium of CFT stuff  $\Leftrightarrow$  AdS black hole

$T \Leftrightarrow$  location of horizon  $r = r_H$

Local thermal equilibrium (hydro)  $\Leftrightarrow$  slowly-varying deformations of

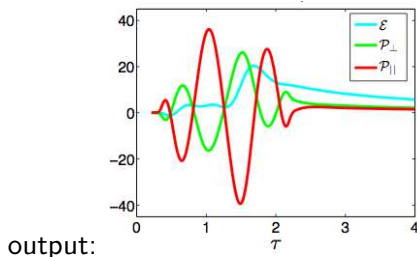
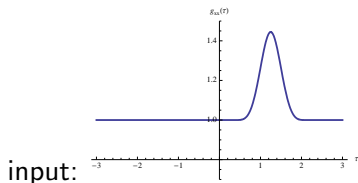
AdS BH:  $r = r_H(\vec{x}, t)$ . [Janik-Peschanski, Bhattacharyya et al]

# Approach to equilibrium

bulk picture: dynamics of gravitational collapse.

dissipation: energy falls into BH [Horowitz-Hubeny, 99]

- quasinormal modes of a small BH [Freiss et al, 06]  $\tau_{th} \sim \frac{1}{8T_{peak}}$ .
- far-from equilibrium processes: [Chesler-Yaffe, 08, 09] (PDEs!)



black hole forms from vacuum initial conditions.

brutally brief summary: all relaxation timescales  $\tau_{th} \sim T^{-1}$ .

- Lesson: In these models, breakdown of hydro in this model is not set by higher-derivative terms, but from non-hydrodynamic modes.

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Basic question for the holographic description:

How to make a finite density of fermions?

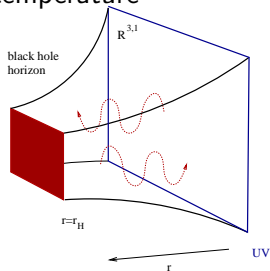
# Minimal ingredients for gravity dual of NFL:

metric  $g_{\mu\nu}$  in bulk  $\Leftrightarrow$  any QFT has a stress tensor  $T^{\mu\nu}$

gauge field  $A_\mu$  in bulk  $\Leftrightarrow$  proxy for fermion number global current  $j^\mu$

bulk spinor field  $\psi \Leftrightarrow$  proxy for bare electrons fermionic operator  $\Psi$

black hole (BH) in bulk  $\Leftrightarrow$  finite temperature



charged BH  $\Leftrightarrow$  finite charge density

# black hole solution

Consider any relativistic CFT with a gravity dual and a conserved  $U(1)$  symmetry. Any  $d > 1 + 1$ , focus on  $d = 2 + 1$ .

The gravity dual is described by:

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{g} \left( \mathcal{R} + \frac{6}{R^2} - \frac{2\kappa^2}{g_F^2} F_{\mu\nu} F^{\mu\nu} + \dots \right)$$

(... = fields which vanish in groundstate, more irrelevant couplings.)

Solution describing a finite density of  $U(1)$  charge: (focus on  $T = 0$ .)

$$ds^2 = \frac{r^2}{R^2} (-f dt^2 + d\vec{x}^2) + R^2 \frac{dr^2}{r^2 f}, \quad A = \mu \left( 1 - \frac{1}{r} \right) dt$$

$$f(r) = 1 + \frac{3}{r^4} - \frac{4}{r^3}$$

$\mu$  = chemical potential.      horizon at  $r = 1$ .

## a crossroads

Entropy density of black hole:

$$s(T=0) = \frac{1}{V_{d-1}} \frac{A}{4G_N} = 2\pi e_d \rho. \quad (e_d \equiv \frac{g_F}{\sqrt{2d(d-1)}})$$

This is a large low-energy density of states!

not supersymmetric ... lifted at finite  $N$

## a crossroads

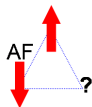
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**pessimism:**  $S(T=0) \neq 0$  violates third law of thermodynamics, unphysical, weird string-theorist nonsense.



[Wikipedia]

**optimism:** this is a model of frustration:

large degeneracy in classical limit

quantumly, very many states whose energy differences are parametrically smaller than some UV scale ( $\mu$ ).

# Strategy to find a Fermi surface

Look for sharp features in fermionic Green functions  
at finite momentum and small frequency [S-S Lee].

Assume that amongst the ... in the bulk action is

$$S_{\text{probe}}[\Psi] = \int d^{D+1}x \sqrt{g} (\bar{\Psi} (\not{D} - m) \Psi + \text{interactions})$$

with  $D_{\mu}\psi = \partial_{\mu}\psi - iqA_{\mu}\psi$

$$\Delta = \frac{d}{2} \pm mR, \quad q = q$$

'Bulk universality': for two-point functions, the interaction terms don't matter!

Results only depend on  $q, \Delta$ .

## Comments about the strategy

- ▶ There are many string theory vacua with these ingredients. In specific examples of dual pairs (e.g. M2-branes  $\Leftrightarrow$  M th on  $AdS_4 \times S^7$ ), interactions and  $\{q, m\}$  are specified.  
which sets  $\{q, m\}$  are possible and what correlations there are is not clear.
- ▶ This is a large complicated system ( $\rho \sim N^2$ ), of which we are probing a tiny part ( $\rho_\Psi \sim N^0$ ).
- ▶ It would be surprising if we could describe a Fermi liquid (= a weakly coupled QFT).
- ▶ In general, both bosons and fermions of the dual field theory are charged under the  $U(1)$  current: this is a Bose-Fermi mixture.

Let's see what happens.

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## AdS/CFT prescription for spinors

To compute  $G_R$ : solve Dirac equation in BH geometry, impose infalling boundary conditions at horizon [Son-Starinets, Iqbal-Liu].

Like retarded response, falling into the BH is something that *happens*.

Translation invariance in  $\vec{x}, t \implies$  ODE in  $r$ .

Rotation invariance:  $k_j = \delta_j^1 k$

Near the boundary, solutions behave as

$$\psi^{r \rightarrow \infty} \approx a_\alpha r^m \begin{pmatrix} 0 \\ 1 \end{pmatrix} + b_\alpha r^{-m} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

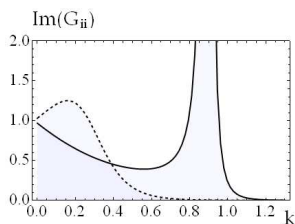
Matrix of Green's functions, has two independent eigenvalues:

$$G_\alpha(\omega, \vec{k}) = \frac{b_\alpha}{a_\alpha}, \quad \alpha = 1, 2$$

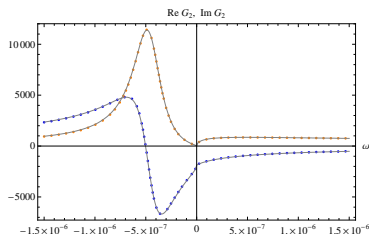
Eqn depends on  $q$  and  $\mu$  only through  $\mu_q \equiv \mu q$

$\rightarrow \omega$  is measured from the effective chemical potential,  $\mu_q$ .

# Fermi surface!



MDC:  $G(\omega = -0.001, k)$



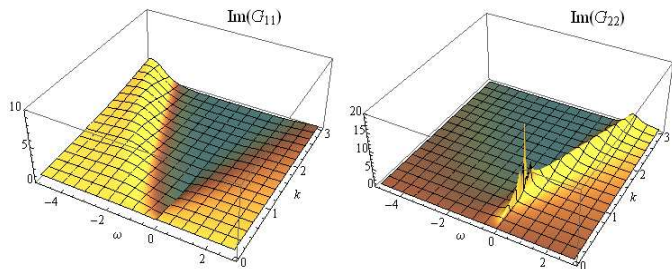
EDC:  $G(\omega, k = 0.9)$

$$k_F \approx 0.918528499$$

Basic checks:

- $\text{Im } G \geq 0$ ,  $\forall \omega$  (unitarity)
- the only non-analyticity is at  $\omega = 0$
- approaches CFT behavior for  $|\omega| \gg \mu$

# It's not a Fermi liquid



The peak moves with dispersion relation  $\omega \sim k_{\perp}^z$  with

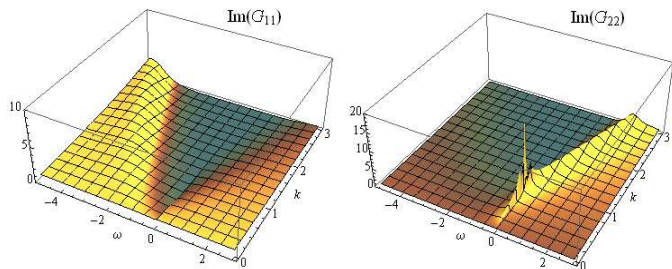
$$z = 2.09 \text{ for } q = 1, \Delta = 3/2.$$

$$z = 5.32 \text{ for } q = 0.6, \Delta = 3/2.$$

Scaling behavior:  $G_R(\lambda k_{\perp}, \lambda^z \omega) = \lambda^{-\alpha} G_R(k_{\perp}, \omega)$ ,  $\alpha = 1$ .

vs Landau Fermi liquid:  $z = \alpha = 1$

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Whence these exponents?

Take apart the black box.

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## Emergent quantum criticality

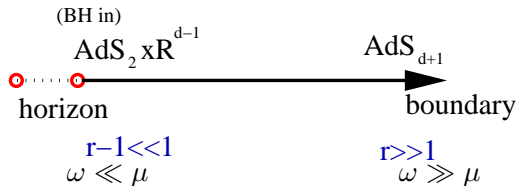
$$ds^2 = \frac{r^2}{R^2} (-fdt^2 + d\vec{x}^2) + R^2 \frac{dr^2}{r^2 f}$$

At  $T = 0$ :  $f \approx 6(r - 1)^2$

$\implies$  near-horizon geometry is  $AdS_2 \times \mathbb{R}^{d-1}$

The conformal invariance of this metric is **emergent**.

(We broke the microscopic conformal invariance with finite density.)



AdS/CFT suggests that the low-energy physics is governed by the dual **IR CFT**.

(A  $CFT_0$  (or chiral  $CFT_1$ ) for each  $\vec{k}$ .)

The bulk geometry is a picture of the RG flow from the  $CFT_d$  to this NRCFT.

# Low-frequency behavior

$T > 0$ :  $G_R(\omega)$  analytic near  $\omega = 0 \rightarrow$  can compute in series expansion. [Policastro-Son-Starinets]

$T = 0$ : Expanding the wave equation in  $\omega$  is delicate.

The  $\omega$ -term dominates near the horizon.

Method of matched asymptotic expansions:

Find solution (in  $\omega$ -expansion) in two regions of BH geometry (IR and UV), match their behavior in the region of overlap.

Familiar from brane absorption calculations.

Here: this 'matching' can be interpreted in the QFT as RG matching between UV and IR CFTs.

(We need only assume the existence of the IR CFT, the gravity dual lets us compute.)

## Inner region (IR data)

Wave equations for charged fields in  $AdS_2$  are solvable.  
Near the boundary:

$$\psi \stackrel{\sigma \rightarrow 0}{\approx} \mathcal{G} \sigma^\nu v_+ + \sigma^{-\nu} v_-$$

$$\nu_k \equiv R_2 \sqrt{m^2 + k^2 - q^2/2}, \quad \delta_k = \frac{1}{2} + \nu_k$$

For a spinor in  $AdS_2$ ,  $k$  is a parity-violating mass term  $\tilde{m} \bar{\psi} \Gamma \psi$ :  $\tilde{m} \equiv k \frac{R_2}{r_0}$   
 $\Psi(\omega, k)$  matches onto some IR CFT operator  $\mathcal{O}_k$  of dimension  
 $\delta_k = \frac{1}{2} + \nu_k$ , whose two-point function is the

$$\text{IR CFT Green function: } \mathcal{G}_k(\omega) = c(k) \omega^{2\nu_k}$$

$c(k) \in \mathbb{C}$ , known.

# Low-frequency expansion in outer region (UV data)

Basis of solutions at  $\omega = 0$ :

$$\psi_{\alpha}^{(0)\pm} \stackrel{r \rightarrow 1}{\approx} v_{\pm} (r-1)^{\mp\nu}$$

These two solutions match to the leading and subleading solutions in the near-horizon region.

$$\implies \psi_{\alpha} = \psi_{\alpha}^{+} + \mathcal{G}(\omega) \psi_{\alpha}^{-}.$$

$$\psi_{\alpha}^{\pm} = \psi_{\alpha}^{(0)\pm} + \omega \psi_{\alpha}^{(1)\pm} + \omega^2 \psi_{\alpha}^{(2)\pm} + \dots, \quad \psi_{\alpha}^{(n)\pm} \stackrel{r \rightarrow \infty}{\approx} \begin{pmatrix} b_{\alpha}^{(n)\pm} r^{-m} \\ a_{\alpha}^{(n)\pm} r^m \end{pmatrix}.$$

$$G_R(\omega, k) = K \frac{b_{+}^{(0)} + \omega b_{+}^{(1)} + O(\omega^2) + \mathcal{G}_k(\omega) \left( b_{-}^{(0)} + \omega b_{-}^{(1)} + O(\omega^2) \right)}{a_{+}^{(0)} + \omega a_{+}^{(1)} + O(\omega^2) + \mathcal{G}_k(\omega) \left( a_{-}^{(0)} + \omega a_{-}^{(1)} + O(\omega^2) \right)}$$

## Away from Fermi surface

$$G_R(\omega, k) = K \frac{b_+^{(0)} + \omega b_+^{(1)} + O(\omega^2) + \mathcal{G}_k(\omega) \left( b_-^{(0)} + \omega b_-^{(1)} + O(\omega^2) \right)}{a_+^{(0)} + \omega a_+^{(1)} + O(\omega^2) + \mathcal{G}_k(\omega) \left( a_-^{(0)} + \omega a_-^{(1)} + O(\omega^2) \right)}$$

For generic  $k$ ,  $a_+^{(0)}(k) \neq 0$ .

$$G_R(\omega, k) = \frac{b_+^{(0)}}{a_+^{(0)}} + r_1 \omega + r_2 \mathcal{G}_k(\omega) + \dots$$

$$\mathcal{G}_k(\omega) = c(k) \omega^{2\nu_k}$$

$a_{\pm}, b_{\pm}$  are all real  $\implies$

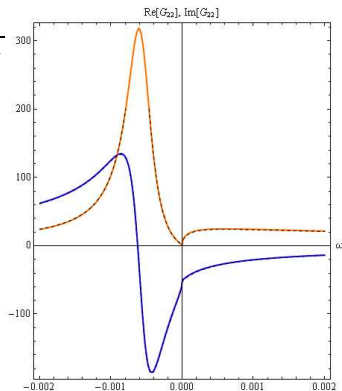
Nonanalytic behavior and dissipation controlled by IR CFT.

# Consequences for Fermi surface

$$G_R(\omega, k) = K \frac{b_+^{(0)} + \omega b_+^{(1)} + O(\omega^2) + \mathcal{G}_k(\omega) \left( b_-^{(0)} + \omega b_-^{(1)} + O(\omega^2) \right)}{a_+^{(0)} + \omega a_+^{(1)} + O(\omega^2) + \mathcal{G}_k(\omega) \left( a_-^{(0)} + \omega a_-^{(1)} + O(\omega^2) \right)}$$

Suppose  $a_+^{(0)}(k = k_F) = 0$ . (Zero-energy boundstate of outer-region eqn.)

$$G_R(\omega, k) = \frac{h_1}{k_\perp - \frac{1}{v_F}\omega - h_2 c(k) \omega^{2\nu_{k_F}}}$$



Correctly fits numerics near FS:

$$(k = .9, m = 0, q = 1 \implies \nu_{k_F} = \frac{1}{\sqrt{6}} \sqrt{m^2 + k_F^2 - q^2/2} = 1.045\dots \geq \frac{1}{2})$$

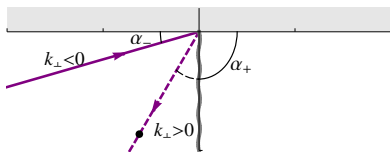
## Relevant operator: singular FL

Suppose  $\frac{1}{2} > \nu_{k_F} \equiv \sqrt{m^2 + k_F^2} - q^2/2/\sqrt{6}$ :

→  $\mathcal{O}_{k_F}$  is **relevant**      $\delta_k = \frac{1}{2} + \nu_k < 1$ .

$$G_R(\omega, k) = \frac{h_1}{k_{\perp} + \frac{1}{v_F}\omega + c_k \omega^{2\nu_{k_F}}}$$

$$\omega_*(k) \sim k_{\perp}^z, \quad z = \frac{1}{2\nu_{k_F}} > 1$$



$$\frac{\Gamma(k)}{\omega_*(k)} \xrightarrow{k_{\perp} \rightarrow 0} \text{const}, \quad Z \propto k_{\perp}^{\frac{1-2\nu_{k_F}}{2\nu_{k_F}}} \xrightarrow{k_{\perp} \rightarrow 0} 0.$$

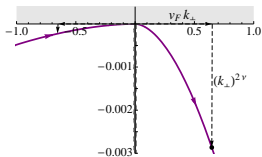
Not a stable quasiparticle.

## Irrelevant operator: Fermi liquid

Suppose  $\nu_{k_F} > \frac{1}{2}$ :  $\rightarrow \mathcal{O}_{k_F}$  is **irrelevant**  $\delta_k = \frac{1}{2} + \nu_k > 1$ .

$$G_R(\omega, k) = \frac{h_1}{k_{\perp} + \frac{1}{v_F}\omega + c_k \omega^{2\nu_{k_F}}}$$

$$\omega_{\star}(k) \sim v_F k_{\perp}$$



$$\frac{\Gamma(k)}{\omega_{\star}(k)} \propto k_{\perp}^{2\nu_{k_F}-1} \quad k_{\perp} \rightarrow 0 \rightarrow 0 \quad Z \xrightarrow{k_{\perp} \rightarrow 0} h_1 v_F.$$

A stable quasiparticle, but **never Landau Fermi liquid**.  
(different thermo, transport.)

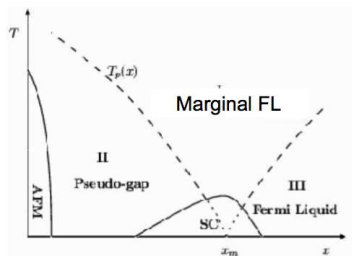
# Marginal operator: “Marginal Fermi liquid”

Suppose  $\nu_{k_F} = \frac{1}{2}$ :  $\mathcal{O}_{k_F}$  is *marginal*.  $\delta_k = \frac{1}{2} + \nu_k = 1$ .  
 $\nu_F \rightarrow 0$ ,  $c(k_F)$  has a pole.

$$G_R \approx \frac{h_1}{k_{\perp} + \tilde{c}_1 \omega \ln \omega + c_1 \omega}, \quad \tilde{c}_1 \in \mathbb{R}, \quad c_1 \in \mathbb{C}$$

$$Z \sim \frac{1}{|\ln \omega_*|} \xrightarrow{k_{\perp} \rightarrow 0} 0.$$

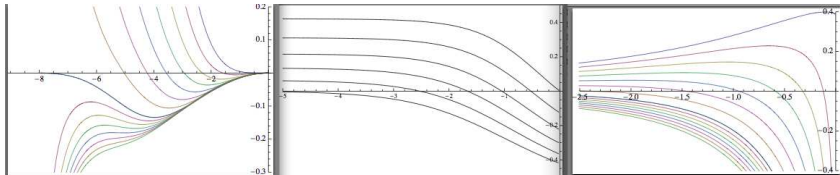
A well-named phenomenological model of high- $T_c$  cuprates near optimal doping



[Varma et al, 1989].

## UV data: where are the Fermi surfaces?

Above we assumed  $a(k_F)_+^{(0)} = 0$ . This happens at  $k_F$ :  $k$  s.t.  $\exists$  normalizable, incoming solutions at  $\omega = 0$ .



Schrodinger potential  $V(\tau)/k^2$  at  $\omega = 0$  for  $m < 0, m = 0, m > 0$ .

$\tau$  is the tortoise coordinate

Right ( $\tau = 0$ ) is boundary; left is horizon.

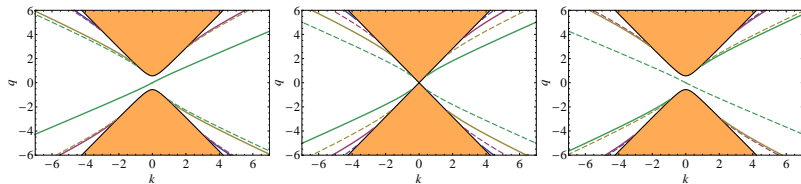
$k > qe_d$ : Potential is always positive

$k < k_{osc} \equiv \sqrt{(qe_d)^2 - m^2}$ : near the horizon  $V(x) = \frac{\alpha}{\tau^2}$ , with

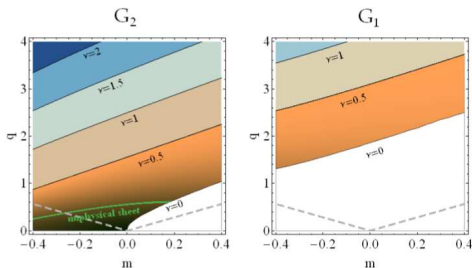
$\alpha < -\frac{1}{4}$  ("oscillatory region")

$k \in (qe_d, k_{osc})$ : the potential develops a potential well, indicating possible existence of a zero energy bound state.

# Where are the Fermi surfaces?



$$\delta_k = \frac{1}{2} + \nu_k, \quad \nu_k = \frac{1}{\sqrt{6}} \sqrt{m^2 + k^2 - q^2/2}$$



## Other interesting things I have to skip

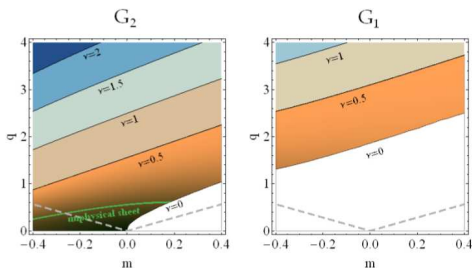
- ▶ particle-hole asymmetry
- ▶ disappearance of FS under relevant deformation of UV CFT
- ▶ if  $\nu \in \mathbb{C}$   $\longrightarrow$   $G$  is log-periodic in frequency
- ▶ formula for Fermi velocity
- ▶ free-fermion limit
- ▶ statistics and stability
- ▶ finite temperature

# Summary

The location of the Fermi surface is determined by short-distance physics (find normalizable sol'n of  $\omega = 0$  Dirac equation in full BH – analogous to band structure)

but the low-frequency scaling behavior near the FS is universal (determined by near-horizon region – IR CFT  $\mathcal{G}$ )

Depending on the dimension of the operator in the IR CFT, we find Fermi liquid behavior (but not Landau) or non-Fermi liquid behavior:



# Outline

## Introduction: Fermions at finite density

Theoretical ubiquity of Landau fermi liquid

Existence of non-Fermi liquids

## Holographic duality

Lightning review

Sample application

## Charged AdS black holes and frustration

## Fermion green functions, numerically

## Analytic understanding of Fermi surface behavior

## Concluding remarks

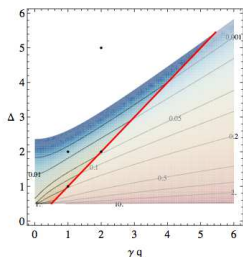
## Two questions

1. **Third law:** Can we get this NFL behavior without large low-energy density of states?

Presumably: Small-freq behavior depended on existence of IR CFT, not large  $c \propto s(T=0)$  of IR CFT.

2. **Charged bosons:** In many explicit dual pairs,  $\exists$  charged scalars.

- If their mass/charge is big enough, they don't condense:



[Denef-Hartnoll]

(vs: a weakly-coupled charged boson at  $\mu \neq 0$  will condense.)

Like moduli stabilization.

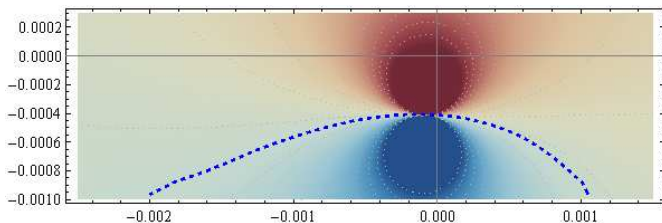
- Many systems to which we'd like to apply this also have a superconducting region.

## Concluding remarks

1. The green's function near the FS is of the form found previously in perturbative calculations, but the nonanalyticity can be order one.
2. The knowledge of quantum statistics displayed by the classical wave equations is remarkable and necessary for AdS/CFT to be consistent with basic facts about many-body physics.
3. [Deneff-Hartnoll-Sachdev] The leading  $N^{-1}$  correction exhibits quantum oscillations.
4. Next: thermodynamics and transport.

the end

## finite temperature



The complex omega plane for  $T = 4.13 \times 10^{-4}$ :  
the quasi-particle pole is a finite distance *below* the real  $\omega$ -axis.  
dashed line: trajectory of the pole between  
 $k = 0.87$ (left) . . .  $0.93$ (right).  $\min_k (\text{Im}\omega_c) \simeq T$  (up to 1% accuracy).  
In background: density plot for  $\text{Im} G_{22}(\omega)$  at  $k = 0.90$   
where the corresponding pole is closest to the real axis.

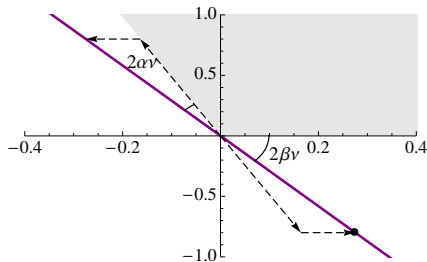
There is a numerical instability for  $\text{Im}\omega < -\pi T$   
(can also be seen directly from the wave equation:  
the outgoing solution near the horizon dominates exponentially over the desired  
incoming solution.)

# Fermion poles always in LHP!

$$\arg c_k = \arg (e^{2\pi i\nu} \pm e^{-2\pi qe_d}) \quad \mathcal{G} = c_k \omega^{2\nu}$$

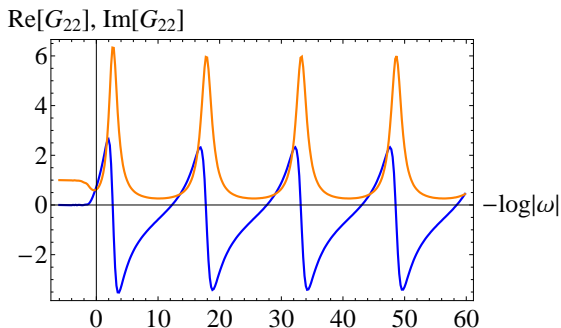
$\pm$  for boson/fermion.

$$\omega_c^{2\nu} = \text{real} \cdot (e^{-2\pi i\nu} - e^{-2\pi qe_d}).$$



**Figure:** A geometric argument that poles of the fermion Green function always appear in the lower-half  $\omega$ -plane: Depicted here is the  $\omega^{2\nu}$  covering space on which the Green function is single-valued. The shaded region is the image of the upper-half  $\omega$ -plane of the physical sheet.

## oscillatory region and log-periodicity



**Figure:** Both  $\text{Re } G_{22}(\omega, k = 0.5)$  (blue curve) and  $\text{Im } G_{22}(\omega, k = 0.5)$  (orange) are periodic in  $\log \omega$  as  $\omega \rightarrow 0$ .

## fermi velocity

Think of  $\omega = 0$  Dirac eqn as Schrödinger problem.

Like Feynman-Hellmann theorem:  $\partial_k \langle H \rangle = \langle \partial_k H \rangle$

we can derive a formula for  $v_F$  in terms of expectation values in the bound-state wavefunction  $\Phi_{(0)}^+$ .

Let:

$$\langle \mathcal{O} \rangle \equiv \int_{r_*}^{\infty} dr \sqrt{g_{rr}} \mathcal{O} ,$$

$$J^\mu \equiv \bar{\Phi}_{(0)}^+ \partial_{k_\mu} \mathcal{D}_{0,k_F} \Phi_{(0)}^+ = \bar{\Phi}_{(0)}^+ \Gamma^\mu \Phi_{(0)}^+$$

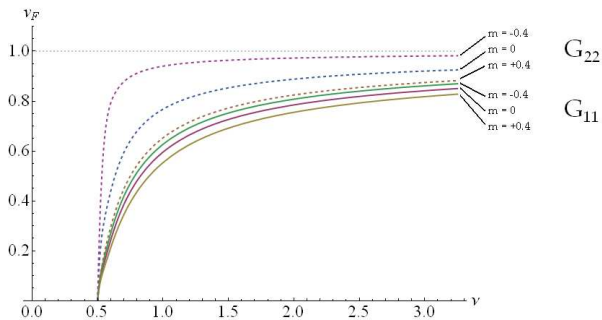
is the bulk particle-number current.

$$v_F = \frac{\langle J^1 \rangle}{\langle J^0 \rangle} = \frac{\int dr \sqrt{g_{rr} g^{ii}} (|y|^2 - |z|^2)}{\int dr \sqrt{g_{rr} (-g^{tt})} (|y|^2 + |z|^2)} .$$

$$\Phi = \begin{pmatrix} y \\ z \end{pmatrix}$$

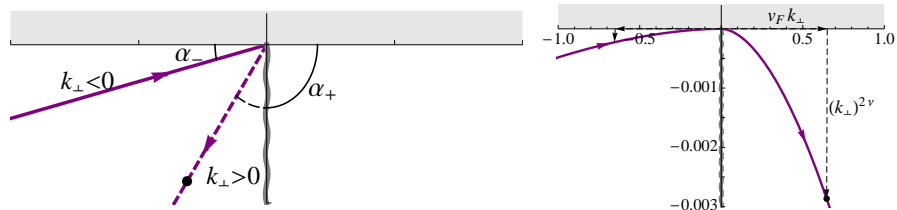
Note:  $\frac{g^{ii}}{-g^{tt}} = f(r) \leq 1$  implies that  $v_F \leq c$ .

# fermi velocity



**Figure:** The Fermi velocity of the primary Fermi surface of various components as a function of  $2\nu > 1$  for various values of  $m$ .

# An explanation for the particle-hole symmetry



**Figure:** Left: Motion of poles in the  $\nu < \frac{1}{2}$  regime. As  $k$  varies towards  $k_F$ , the pole moves in a straight line (hence  $\Gamma \sim \omega_c$ ), and hits the branch point at the origin at  $k = k_F$ . After that, depending on  $\gamma(k_F)$ , it may move to another Riemann sheet of the  $\omega$ -plane, as depicted here. In that case, no resonance will be visible in the spectral weight for  $k > k_F$ . Right: Motion of poles in the  $\nu > \frac{1}{2}$  regime, which is more like a Fermi liquid in that the dispersion is linear in  $k_{\perp}$ ; the lifetime is still never of the Landau form.

**Note:** the location of the branch cut is determined by physics: at  $T > 0$ , it is resolved to a line of poles.

## Oscillatory region

Above we assumed  $\nu = R_2 \sqrt{m^2 + k^2 - (qe_d)^2} \in \mathbb{R}$

$$\nu = i\lambda \Leftrightarrow \text{Oscillatory region.}$$

This is when particle production occurs in  $AdS_2$ . [Pioline-Troost]

Effective mass below BF bound *in*  $AdS_2$ . [Hartnoll-Herzog-Horowitz]

$\text{Re} \omega^{j2\lambda} = \sin 2\lambda \log \omega \implies$  periodic in  $\log \omega$  with period  $\frac{\pi}{|\nu|}$ .

comments about boson case:

Net flux into the outer region  $> 0$  = superradiance of AdS RN black hole (rotating brane solution in 10d)

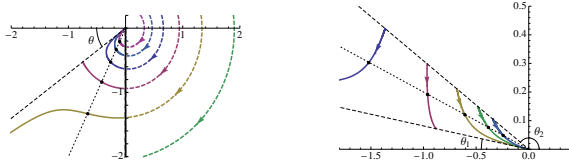
Classical equations know quantum statistics!

like: statistics functions in greybody factors

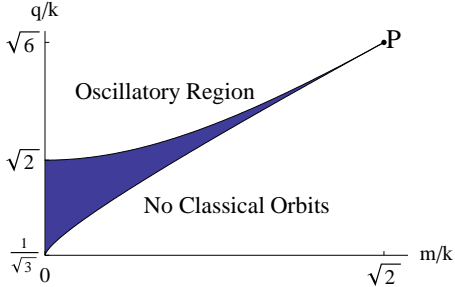
Required for consistency of AdS/CFT!

boson: particles emitted from near-horizon region, bounce off  $AdS_{d+1}$  boundary and return, causing further stimulated emission.

spinor: there is particle production in  $AdS_2$  region, but net flux into the outer region is negative ('no superradiance for spinors').



**Figure:** The motion of poles of the Green functions of spinors (left) and scalars (right) in the complex frequency plane. Both plots are for parameter values in the oscillatory region ( $q = 1, m = 0$ ). In order to give a better global picture, the coordinate used on the complex frequency plane is  $s = |\omega|^{\frac{1}{20}} \exp(i \arg(\omega))$ . The dotted line intersects the locations of the poles at  $k = k_0 = \dots$ , and its angle with respect to the real axis is determined by  $\mathcal{G}(k, \omega)$ . The dashed lines in the left figure indicate the motion of poles on another sheet of the complex frequency plane at smaller values of  $k < k_0$ . As  $k$  approaches the boundary of the oscillatory region, most of the poles join the branch cut. It seems that one pole that becomes the Fermi surface actually manages to stay in place. These plots are only to be trusted near  $\omega = 0$ .



**Figure:** Information from WKB. At large  $q$ ,  $m$ , the primary Fermi momentum is given by the WKB quantization formula:

$k_F \int_{s_-}^{s_+} ds \sqrt{V(s; \alpha, \beta)} = \pi$ , where  $\alpha \equiv \frac{q}{k}$ ,  $\beta \equiv \frac{m}{k}$ ,  $s$  is the tortoise coordinate, and  $s_{\pm}$  are turning points surrounding the classically-allowed region. For  $k < q/\sqrt{3}$ , the potential is everywhere positive, and hence there is no zero-energy boundstate. This line intersects the boundary of the oscillatory region at  $k^2 + m^2 = q^2/2$  at the point  $P = (\alpha, \beta) = (\sqrt{6}, \sqrt{2})$ . Hence, only in the shaded (blue) region is there a Fermi surface. The exponent  $\nu(k_F)$  is then given by

$\nu(k_F) = \frac{\pi \sqrt{1 + \beta^2 - \alpha^2/2}}{\int ds \sqrt{V(s; \alpha, \beta)}}$ . This becomes ill-defined at the point  $P$ , and interpolates between  $\nu = 0$  at the boundary of the oscillatory region, and

$\nu = \infty$  at  $k = q/\sqrt{3}$ .

## AdS/CFT prescription for spinors

$$\Delta = \frac{d}{2} \pm mR$$

$$\Gamma^a e_a{}^M \left( \partial_M + \frac{1}{4} \omega_{abM} \Gamma^{ab} - iqA_M \right) \psi - m\psi = 0$$

Rotation invariance:  $k_j = \delta_j^1 k$

$$\Phi_\alpha \equiv (-g g^{rr})^{-1/4} \Pi_\alpha^{\hat{k}} \psi, \quad \psi = e^{-i\omega t + ik_i x^i} \psi_{\omega, k},$$

$$\boxed{(\partial_r + M\sigma^3) \Phi_\alpha = ((-1)^\alpha K\sigma^1 + W i\sigma^2) \Phi_\alpha, \quad \alpha = 1, 2}$$

with

$$M \equiv m\sqrt{g_{rr}} = \frac{m}{r\sqrt{f}}, \quad K \equiv k\sqrt{\frac{g_{rr}}{g_{ii}}} = \frac{k}{r^2\sqrt{f}}, \quad W \equiv u\sqrt{\frac{g_{rr}}{g_{ii}}} = \frac{u}{r^2\sqrt{f}}.$$

$$u \equiv \sqrt{\frac{-g^{tt}}{g^{ii}}} \left( \omega + \mu_q \left( 1 - \left( \frac{r_0}{r} \right)^{d-2} \right) \right)$$

## relations satisfied by the spinor Green's functions

$$G_2(\omega, k) = G_1(\omega, -k) \quad \longrightarrow k > 0 .$$

$$G_1(\omega, k; -q) = -G_2^*(-\omega, k; q) \quad \longrightarrow q > 0 .$$

$$G_2(\omega, k; m) = -\frac{1}{G_1(\omega, k; -m)} \quad \longrightarrow m > 0 .$$

When  $m \in [0, \frac{1}{2})$  there is an alternative quantization where

$$\tilde{G}_\alpha = -\frac{a_\alpha}{b_\alpha} .$$